The Conjugate of an operator

Let N be a NLS.

$N_s \rightarrow$ linear space of all scalar valued fns defined on N.

$N^* \rightarrow$ linear space of all continuous L.T. defined ~~on N~~ from N into $\mathbb{R}$ or $\mathbb{C}$

Then N^* is a linear subspace of N_s.

Conjugate (or Adjoint) of a L.T. $T \rightarrow$

Let N be NLS. T be an operator on N i.e. T be a continuous L.T. of N into itself. Define a L.T. T^* of N^* into itself as follows →

If $f \in N^*$ then $T^*(f)$ is given by—

$$[T^*(f)](x) = f(T(x))$$

Then T^* is called the conjugate (or adjoint) of T.

Note → T^* is linear as →

$$[T^*(\alpha f + \beta g)](x) = [\alpha T^*(f) + \beta T^*(g)](x)$$

so $T^*(\alpha f + \beta g) = \alpha T^*(f) + \beta T^*(g)$

Thm Let T be an operator on a NLS N then its conjugate T^*, defined by —

$$T^* : N^* \to N^*$$

$$\text{s.t. } T^*(f) = f \circ T$$

and $[T^*(f)](x) = f(T(x)) \quad \to (1)$

for all $f \in N^*$ & $\forall x \in N$,

is an operator on N^*.

Also the mapping →

$$\phi : B(N) \to B(N^*)$$

$$\text{s.t. } \phi(T) = T^* \quad \forall \; T \in B(N)$$

is an isometric isomorphism of $B(N)$ into $B(N^*)$ which reserves products and preserves the identity transformation.

Pf→ We know that T^* is linear.

To show that T^* is an operator on N, we shall show that T^* is bdd.

we have→

$$\|T^*\| = \sup\{\|T^*(f)\| \mid \|f\| \leq 1\}$$

$$= \sup\{\|[T^*(f)](x)\| \mid \|f\| \leq 1, \|x\| \leq 1\}$$

$$= \sup\{\|f(T(x))\| \mid \|f\| \leq 1, \|x\| \leq 1\}, \text{ from eqn (1)}$$

$$\leq \sup\{\|f\| \, \|T\| \, \|x\| \mid \|f\| \leq 1, \|x\| \leq 1\}$$

$$\leq \|T\| \quad \to (2)$$

Since T is bdd. ∴ from (2), T^* is also bdd.

$\therefore$ T^* is an operator on N^*.

Then for each non-zero vector x in N, $\exists$ a functional $f \in N^*$ s.t. $\|f\| = 1$

and $f(T(x)) = \|T(x)\| \longrightarrow (3)$

$$\therefore \|T\| = \sup \left\{ \frac{\|T(x)\|}{\|x\|} \;\middle|\; x \neq 0 \right\}$$

$$= \sup \left\{ \frac{\|f(T(x))\|}{\|x\|} \;\middle|\; \|f\| = 1, x \neq 0 \right\}$$

, from eqn (3)

$$= \sup \left\{ \frac{\|T^*(f)(x)\|}{\|x\|} \;\middle|\; \|f\| = 1, x \neq 0 \right\}$$

, from (1)

$$\leq \sup \left\{ \frac{\|T^*(f)\|\,\|x\|}{\|x\|} \;\middle|\; \|f\| = 1, x \neq 0 \right\}$$

$$\neq \sup \left\{ \|T^*(f)\| \;\middle|\; \|f\| = 1 \right\}$$

$\|T\| \leq \|T^*\|$, by defn of norm of T^* w.r.t. T.

$\rightarrow (4)$

$\therefore$ from (2) & (4), we get $\rightarrow$

$\|T^*\| = \|T\| \longrightarrow (5)$

(Contd.)

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